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STRESS FIELDS IN COMPOSITE LAMINATES

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Abstract—A new theory is proposed to define the complete stress field within an arbitrary composite laminate. The theory is based upon an extension of Reissner's variational principle to laminated bodies. Weaknesses in previous laminate theories are discussed and it is demonstrated how these are overcome in the present formulation. Comparison with existing numerical elasticity solutions for a class of boundary value problems in which steep stress gradients are present shows extremely close agreement.

INTRODUCTION

In the contemporary technology of structural composite materials, major deficiencies exist with respect to our ability to determine the stress field within a multilayered composite laminate. In most cases, even a superficial or qualitative understanding of the nature of the stress field in regions of steep stress gradients has not been established. A notable exception is the approximate treatment[1, 2] of the classical free edge problem in laminate elasticity[3, 4], however, similar treatments have not been advanced for other laminate stress concentration problems. In the absence of a practical means of laminate stress analysis, it is not possible to develop an understanding and general characterization of the various insidious failure modes which been demonstrated in composite laminates[2, 5–8]. The latter references all document heterogeneous damage development which varies through the laminate thickness and suggest the importance of defining the stress field within each layer, particularly in regions of stress concentration, where classical lamination theory[9, 10] errs badly[4, 11, 12]. This is contrasted to the *ultimate* failures of certain laminates in the presence of stress risers, which only seem to depend upon the overall laminate properties[13, 14], rather than the details of the stacking sequence, although the details of the damage development and growth in these laminates are a function of stresses in the individual layers[15].

Although the finite element method is widely used in the design of practical composite structural elements, e.g. [16], its application is limited to determination of force (per unit length), rather than stress distributions. This is accomplished through the assumption of a simplified displacement field-that which is assumed in classical lamination theory. This permits one to define effective elastic properties of the laminate as a whole, and to compute integrated values of the in-plane stress components across the laminate thickness. In fact, once the force distribution has been defined, the associated layer stresses may be computed, however, as mentioned earlier, this procedure is not generally reliable. At present, therefore, determinations of failure loading and mode of failure in practical composite structures are based primarily on experimentation with prototype bodies. These comments are not made with the intention of disparaging the contemporary practice since the presence of structural complexity in the form of holes, connections, edges, and discontinuities in thickness of many-layered composites may overwhelm *any* attempted analytical technique.

The basic limitation imposed by numerical solutions to the laminate elasticity problem has been illustrated in a recent paper by Wang and Crossman[17] in their treatment of the free edge class of boundary value problems. In order to achieve a realistic prediction of the stress field by use of the finite element method, sixteen elements in the thickness direction were required within each layer in the region of steep gradients. A total of 196 elements per layer were employed. To accomodate such a large array, it became necessary to employ a special matrix storage scheme for the purpose of reducing computer storage and running time. Similarly, Pipes[18] required a lengthy extrapolation procedure in conjunction with the finite difference method in order to achieve a satisfactory solution for a two layer free edge boundary value problem.

Another approach to the problem of laminate stress analysis, e.g. Rybicki[19], utilizes larger elements that possess a more complex stress field. Stanton *et al.*[20] employ a tricubic isoparametric discrete element and a system to automate the construction of finite element models. The latter approach effects an appreciable reduction in data input requirements. This added efficiency seems to be the major advantage of "large element" formulations, as the number of degrees of freedom is comparable to those employed in the more routine methods. Despite the refinements, however, computed laminate stress fields are not universally accurate, as another feature of laminate elastic analysis, i.e. the presence of stress singularities, is a severe obstacle to the execution of numerical elasticity solutions. Unfortunately, the order of these singularities has not been defined for anisotropic layers, i.e. the extension of Bogy's work on isotropic quarter-planes[21] has not been accomplished.

On the other hand, approximate theories have been proposed in attempts to execute realistic laminate stress analysis. The most popular of these is the aforementioned classical lamination theory, *op. cit.*[9, 10]. While this theory has been shown to yield reasonably accurate stress calculations in certain classes of boundary value problems under a limited range of geometric parameters[22–24], its assumptions are too restrictive for general application. Theories incorporating laminate "shear deformation"[25, 26] lead to accurate deflections in bending problems but offer no hope for improved stress computations[26]. The higher-order plate theory derived by Whitney and Sun[27] was applied by Pagano[28] to examine the interlaminar normal stress distribution in the free edge boundary value problem but only on a plane of symmetry.

The prominent common feature of the theories discussed in the previous paragraph is an assumed displacement field that is continuous across the entire laminate thickness. The theories differ only in their specific choice of the assumed displacements. This displacement assumption, however, guarantees discontinuous tractions at interfaces between layers of different elastic moduli except under elementary loading conditions. Further, the edge (traction) boundary conditions appropriate to this approach are, in general, insufficient to guarantee equilibrium of sub-regions containing the edge boundary under the known (pointwise) tractions, *op. cit.*[28]†. Hence, use of this displacement approach and possible extensions to allow even higher order variations through the thickness, is unacceptable for laminate stress field analysis.

Another class of approximate laminate theories represent attempts to generalize those discussed above and are based upon the assumption that the displacement components are linear functions of z, the thickness coordinate, within each *layer*. In this formalism then, the displacements are piecewise continuous functions. Among the theories which rely on this approach are the so-called effective stiffness theories pioneered by Sun, Achenbach, and Herrmann[29, 30]. Sun and Whitney[31] treated various theories in this class and demonstrated that, given displacement continuity at all interfaces, the number of field equations depend on N (number of layers) only when traction continuity at interfaces is ignored. Otherwise, the number of field equations is only dependent on the generality of the initial assumption, i.e. whether the linear term in z for transverse displacement w is included or dropped. Hence the number of field equations is constant for *all* laminates. Since the same statement can be made with respect to the number of edge boundary conditions, the deficiency of the aforementioned theories[9, 10, 25–27] with respect to sub-region equilibrium applies to the present class as well. The latter theories do, however, yield a more realistic determination of effective laminate dispersion characteristics, which provided the motivation for their development. The assumption of piecewise linear displacements, as well as $w = w(x, y)$, leads to the theory of Srinivas[32], in which the number of field equations and edge boundary conditions does depend upon the number of layers. Therefore, interface traction continuity conditions cannot be satisfied[31]. Furthermore, in this theory, the interlaminar normal stress, which has been shown to be responsible for delamination failures[2], has been neglected. Despite the accurate results obtained for vibration frequencies, deflection, and axial stress in the particular bending problems treated in[32], the theory is not generally applicable for laminate stress analysis.

†In order to guarantee equilibrium of a given sub-region containing the edge boundary, we must have the freedom to prescribe at least 5 traction boundary conditions (3 force components and 2 couples) on its edge. Therefore, if we wish to guarantee equilibrium of each layer of a laminate, we need at least $5N$ edge boundary conditions, where N is the number of layers in the laminate.

Finally, an approach suggested by Pagano[28], i.e. treatment of *each layer* as a plate governed by the Whitney–Sun theory[27], lacks generality since it can be shown that less than $5N$ edge traction boundary conditions are available in that approach. Furthermore, as a consequence of interface displacement continuity, the natural edge traction boundary conditions are coupled, i.e. they involve functions of the tractions acting on two (adjacent) layers. Thus, extension of this approach by allowing higher order displacements is not advisable, since, as in the previous approaches, proper equilibrium of each layer under its prescribed tractions cannot be enforced.

The previous discussion has defined a clear need to examine new approaches for laminate stress analysis. All known approximate laminate theories are based upon assumed displacement fields, which as we have seen, lead to results lacking credibility. In this work therefore, we shall set down requirements to be satisfied by an acceptable laminate field theory and proceed to develop a self-consistent theory in accord with the requirements, which are: (a) All six stress components are non-zero in general. (b) Traction *and* displacement continuity conditions at interfaces between adjacent layers are satisfied. (c) Consider a region within the laminate that is arbitrarily located except that it is bounded by any two of the parallel interfaces of the laminate. We shall require that the computed stress field acting on the surfaces of the arbitrary region, in conjunction with the prescribed traction boundary conditions (pointwise, in the elasticity sense) on those portions of the external laminate boundary which lie in the region, satisfy the conditions of vanishing resultant force and moment identically. Thus, *every* layer must satisfy this requirement, which we shall henceforth refer to as "layer equilibrium". This implies that appropriate force variables in the field theory are force and moment resultants (per unit length) acting on the cross sections of a layer and interlaminar stresses on its interfacial surfaces. Although sub-regions not bounded by interfacial planes need not satisfy layer equilibrium, in problem solving, additional interfaces may be introduced conceptually to improve solution accuracy. In fact, we may view the purpose of this work as an examination of effectiveness of mathematical laminate models in which the response is defined in terms of force and moment resultants and interlaminar stresses.

Although the above requirements do not define a unique theory, we shall treat the simplest theory within this class in the present work. The theory is based upon a variational theorem derived by Reissner[33] and permits the treatment of discontinuous interfaces, i.e. interface cracks. Known solutions for the free edge boundary value problem in laminate elasticity, where pronounced stress gradients occur, will be utilized to assess the consequences of the present concepts.

VARIATIONAL PRINCIPLE FOR LAMINATES

The physical problem of interest in the present work is that of a laminate which is built of anisotropic elastic layers of uniform thickness and is subjected to prescribed tractions and/or displacements on its boundary. The body is bounded by a cylindrical edge surface and upper and lower faces that are parallel to the interfacial planes. Since it is necessary to consider both traction and displacement continuity conditions at the various interfaces, it is logical to examine Reissner's variational theorem[33] as a mechanism to develop the appropriate field equations.

Reissner has shown that the governing equations of elasticity can be obtained as a consequence of the variational equation

$$\delta J = 0 \tag{1}$$

where

$$J = \int_V F \, \mathrm{d}V - \int_{S'} \bar{T}_i u_i \, \mathrm{d}S \tag{2}$$

and

$$F = \frac{1}{2} \sigma_{ij}(u_{i,j} + u_{j,i}) - W. \tag{3}$$

In the above equations, W is the strain energy density expressed in terms of the stresses $\sigma_{ij}(i, j = 1, 2, 3)$, V is the volume, S the entire surface $\bar{\tau}_i$ the prescribed tractions, u_i the displacement components, and S' is the portion of the boundary on which one or more traction components are prescribed. In what follows, we shall let S'' represent the portion of the boundary on which one or more displacement components are prescribed. Summation over the range of repeated subscripts will be understood in this work. It is also understood that both stresses and displacements are subjected to variation in the application of eqn (1).

We shall now express the form of eqn (1) for the laminated body, which is composed of N layers, the volumes of which are represented by $V_k(k = 1, 2, \dots N)$. For conceptual purposes, we may let the layers be numbered consecutively from the bottom ($k = 1$) to the top ($k = N$). Thus we get, by definition

$$J = \sum_{k=1}^{N} \int_{V_k} \left[\frac{1}{2}\sigma_{ij}(u_{i,j} + u_{j,i}) - W\right]^{(k)} \mathrm{d}V_k - \int_{S'} \bar{\tau}_i u_i \, \mathrm{d}S \tag{4}$$

where the superscript (k) attached to the bracket signifies that each variable within the bracket is associated with the kth layer. We shall also incorporate expansional strains e_{ij} [34], or strains produced in the absence of stress, in the present theory, so that

$$W = W(\sigma_{ij}, e_{ij}). \tag{5}$$

Substituting (4) into (1), making some trivial manipulations, and applying the Green–Gauss Theorem gives

$$\sum_{k=1}^{N} \int_{V_k} \left[\left(\frac{u_{i,j} + u_{j,i}}{2} - \frac{\partial W}{\partial \sigma_{ij}}\right) \delta\sigma_{ij} - \sigma_{ij,j}\delta u_i\right]^{(k)} \mathrm{d}V_k - \int_{S'} \bar{\tau}_i \delta u_i \, \mathrm{d}S + \sum_{k=1}^{N} \int_{S_k} \tau_i^{(k)} \delta u_i^{(k)} \, \mathrm{d}S_k = 0 \tag{6}$$

where S_k is the surface enclosing V_k and $\tau_i^{(k)}$ are the tractions components acting on S_k. We should recognize that the surfaces S_k and S_{k+1} contain a common region, namely, the interface between the respective layers. Hence, we shall define surface I_k as the portion of S_k which contains the top of the kth layer, I'_k and I''_k as the regions of I_k belonging to S' and S'' respectively, and I'''_k as the portion of I_k that does not belong to either S' or S''. Observing that the edges of the layers, as well as the top of the Nth layer and bottom of the first layer, all belong to S, eqn (6) may be expressed as

$$\sum_{k=1}^{N} \int_{V_k} \left[\left(\frac{u_{i,j} + u_{j,i}}{2} - \frac{\partial W}{\partial \sigma_{ij}}\right) \delta\sigma_{ij} - \sigma_{ij,j}\delta u_i\right]^{(k)} \mathrm{d}V_k + \int_{S'} (\tau_i - \bar{\tau}_i)\delta u_i \, \mathrm{d}S + \int_{S''} \tau_i \delta u_i \, \mathrm{d}S + \sum_{k=1}^{N-1} \int_{I'''_k} (\tau_i^{(k)}\delta u_i^{(k)} + \tau_i^{(k+1)}\delta u_i^{(k+1)}) \, \mathrm{d}I_k = 0. \tag{7}$$

Clearly, the vanishing of the volume integrals requires satisfaction of the equilibrium equations and stress-displacement relations within each layer. The vanishing of the surface integrals on S' and S'' require that one term of each of the products $(\tau_1 u_1, \tau_2 u_2, \tau_3 u_3)$ be prescribed at each point on S since δu_i is arbitrary on S' and it vanishes on S''. Finally, the integrals over $I'''_k(k = 1, 2, \dots N-1)$ vanish when tractions and displacements are continuous in these regions. Hence, eqn (7), which represents the statement of Reissner's theorem for laminated bodies, will be applied in the derivation of our approximate laminate theory in the next section.

DEVELOPMENT OF THEORY

Consider a single layer of thickness h within the laminate. We let x and y represent the coordinates in the midplane of the layer, which is bounded by the planes $z = \pm h/2$ and the cylindrical edge surface whose intersection with the midplane is called L. The region enclosed by L will be denoted by R. The interlaminar stresses σ_z, τ_{xz} and τ_{yz} at the top of the layer are denoted by p_2, t_2 and s_2, respectively, while the corresponding stresses at the bottom of the layer are designated as p_1, t_1 and s_1. Superscripts (k), which identify the layers, will be dropped except when they are need for clarity.

The simplest assumption consistent with realistic stress analysis is that the in-plane stress components† are linear functions of z, viz.,

$$\begin{aligned} \sigma_1 = \sigma_x &= \frac{N_x}{h} + \frac{12M_x z}{h^3} \\ \sigma_2 = \sigma_y &= \frac{N_y}{h} + \frac{12\,M_y z}{h^3} \\ \sigma_6 = \tau_{xy} &= \frac{N_{xy}}{h} + \frac{12M_{xy} z}{h^3} \end{aligned} \tag{8}$$

where $N_x \ldots M_{xy}$ are functions of x and y only. Obviously, these functions represent the usual force and moment resultants arising in plate theory. We have also indicated the symbols for the stress components in contracted notation since this system will be convenient for future developments.

We now substitute (8), along with the values of the interlaminar stresses at $z = \pm h/2$, into the differential equations of equilibrium, which leads to the following distributions

$$\begin{aligned} \sigma_3 = \sigma_z &= \frac{(p_1+p_2)}{4}\left(\frac{12z^2}{h^2}-1\right) + \frac{(p_2-p_1)}{4}\left(\frac{40z^3}{h^3}-\frac{6z}{h}\right) + \frac{3N_z}{2h}\left(1-\frac{4z^2}{h^2}\right) + \frac{15M_z}{h^2}\left(\frac{2z}{h}-\frac{8z^3}{h^3}\right) \\ \sigma_4 = \tau_{yz} &= (s_2-s_1)\frac{z}{h} + \frac{(s_1+s_2)}{4}\left(12\frac{z^2}{h^2}-1\right) + \frac{3V_y}{2h}\left(1-\frac{4z^2}{h^2}\right) \\ \sigma_5 = \tau_{xz} &= (t_2-t_1)\frac{z}{h} + \frac{(t_1+t_2)}{4}\left(\frac{12z^2}{h^2}-1\right) + \frac{3V_x}{2h}\left(1-\frac{4z^2}{h^2}\right) \end{aligned} \tag{9}$$

where the shear resultants V_x, V_y and the functions N_z, M_z given by

$$(N_z, M_z) = \int_{-h/2}^{h/2} \sigma_z(1, z)\,\mathrm{d}z \tag{10}$$

are functions of x and y alone. The functions on the right hand side of eqns (8) and (9) are not all independent as they are related via equilibrium and continuity considerations, but these relations will be subsequently developed by means of the variational equation (7).

In general, the strain energy density of an elastic anisotropic body is given by

$$W = \frac{1}{2} S_{ij}\sigma_i\sigma_j + \sigma_i e_i \; (i, j = 1, 2 \ldots 6) \tag{11}$$

where contracted notation has been employed, with S_{ij} representing the compliance matrix and e_i the engineering expansional strain components. Since structural composite laminates are generally built such that each layer possesses a plane of elastic symmetry parallel to xy, we shall treat this material class (monoclinic) in this work although generally anisotropic layers may be treated without difficulty. For monoclinic symmetry with respect to the xy plane, the compliance matrix takes the form

$$S_{ij} = \begin{bmatrix} S_{11} & & & & & \\ S_{12} & S_{22} & & & & \\ S_{13} & S_{23} & S_{33} & \text{SYMM.} & & \\ 0 & 0 & 0 & S_{44} & & \\ 0 & 0 & 0 & S_{45} & S_{55} & \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \tag{12}$$

†Note that we refrain from assuming the form of the displacement field in accordance with the objectionable features of that approach described earlier.

while

$$e_4 = 2e_{23} = 0$$
$$e_5 = 2e_{13} = 0 \tag{13}$$

for monoclinic symmetry.

We now substitute eqns (8), (9) and (11), taking account of (12) and (13), into the variational equation (7) and the integration with respect to z is performed. The appropriate field equations and boundary conditions are determined by setting the coefficients of the arbitrary functions (first variations) equal to zero. Only the final results of this lengthy procedure will be shown here. The interested reader is referred to [39], where the complete mathematical details are presented.

In the derivation of the governing equations, the integration with respect to z gives rise to weighted average displacements and displacements at the interfaces. Therefore, we make the definitions

$$(\bar{f}, f^*, \hat{f}) = \int_{-h/2}^{h/2} f\left(1, \frac{2z}{h}, \frac{4z^2}{h^2}\right)\frac{2\,dz}{h} \tag{14}$$

where f may represent either u, v or w, the x, y, z components of displacement, respectively. We also let u_2, v_2, w_2 represent the displacement components at the top of the layer and u_1, v_1, w_1 the corresponding functions at the bottom of the layer. Furthermore, for internal consistency in the theory, we express the prescribed tractions on the appropriate portions of boundary L as follows

$$\begin{aligned}
\tilde{\sigma}_n &= \frac{1}{h}\left(\tilde{N}_n + \frac{12\tilde{M}_n z}{h^2}\right)\\
\tilde{\tau}_{ns} &= \frac{1}{h}\left(\tilde{N}_{ns} + \frac{12\tilde{M}_{ns} z}{h^2}\right)\\
\tilde{\tau}_{nz} &= (\tilde{\tau}_2 - \tilde{\tau}_1)\frac{z}{h} + \frac{(\tilde{\tau}_1 + \tilde{\tau}_2)}{4}\left(\frac{12z^2}{h^2} - 1\right) + \frac{3\tilde{V}_n}{2h}\left(1 - \frac{4z^2}{h^2}\right)
\end{aligned} \tag{15}$$

where n and s are local coordinates, which are respectively normal and tangent to L. We note that $\tilde{\tau}_1$ and $\tilde{\tau}_2$ give the values of shear stress $\tilde{\tau}_{nz}$ at the bottom and top of the layer, respectively. No restrictions are placed on the nature of the boundary tractions and/or displacements over the remainder of the laminate boundary. Finally, the following contractions are introduced

$$\begin{aligned}
Q_4 &= \frac{(4s_1 - s_2)h}{30} - \frac{V_y}{10}\\
Q_5 &= \frac{(4t_1 - t_2)h}{30} - \frac{V_x}{10}\\
T_4 &= \frac{(4s_2 - s_1)h}{30} - \frac{V_y}{10}\\
T_5 &= \frac{(4t_2 - t_1)\,h}{30} - \frac{V_x}{10}\\
\left.\begin{matrix}R_1\\R_2\end{matrix}\right\} &= \frac{(6p_1 + p_2)h^2 - 7N_z h \pm 30M_z}{70h}\\
\left.\begin{matrix}\beta_4\\\alpha_4\end{matrix}\right\} &= \frac{3}{8}h\hat{w}_{,y} - \frac{h}{8}\bar{w}_{,y} - \frac{3}{2}v^* \pm \left(\frac{h}{4}w^*_{,y} - \frac{\bar{v}}{2}\right)\\
\left.\begin{matrix}\beta_5\\\alpha_5\end{matrix}\right\} &= \frac{3}{8}h\hat{w}_{,x} - \frac{h}{8}\bar{w}_{,x} - \frac{3}{2}u^* \pm \left(\frac{h}{4}w^*_{,x} - \frac{\bar{u}}{2}\right)\\
\left.\begin{matrix}\gamma_1\\\gamma_2\end{matrix}\right\} &= -\frac{3}{2}w^* \pm \frac{3}{4}(5\hat{w} - \bar{w})
\end{aligned} \tag{16}$$

Using the above definitions, and letting e_x, e_y, e_z and e_{xy} represent the engineering expansional strain components, we may now record the governing equations in the present theory. The field equations, which consist of the elastic constitutive relations and the differential equations of equilibrium, must be satisfied *within each layer* and are given by:

Constitutive equations

$$
\begin{aligned}
h\left(\frac{\bar{u}_{,x}}{2}-e_x\right) &= S_{11}N_x+S_{12}N_y+S_{13}N_z+S_{16}N_{xy}\\
h\left(\frac{\bar{v}_{,y}}{2}-e_y\right) &= S_{12}N_x+S_{22}N_y+S_{23}N_z+S_{26}N_{xy}\\
3w^*-he_z &= S_{13}N_x+S_{23}N_y+\frac{6}{5}S_{33}N_z+S_{36}N_{xy}-\frac{S_{33}h}{10}(p_1+p_2)\\
h\left(\frac{\bar{u}_{,y}+\bar{v}_{,x}}{2}-e_{xy}\right) &= S_{16}N_x+S_{26}N_y+S_{36}N_z+S_{66}N_{xy}\\
\frac{h^2}{4}u^*_{,x} &= S_{11}M_x+S_{12}M_y+S_{13}M_z+S_{16}M_{xy}\\
\frac{h^2}{4}v^*_{,y} &= S_{12}M_x+S_{22}M_y+S_{23}M_z+S_{26}M_{xy}\\
\frac{5h}{4}(3\hat{w}-\bar{w}) &= S_{13}M_x+S_{23}M_y+\frac{10}{7}S_{33}M_z+S_{36}M_{xy}+\frac{S_{33}h^2}{28}(p_1-p_2)\\
\frac{h^2}{4}(u^*_{,y}+v^*_{,x}) &= S_{16}M_x+S_{26}M_y+S_{36}M_z+S_{66}M_{xy}\\
\frac{3}{4}\left(\bar{w}_{,y}-\hat{w}_{,y}+\frac{4v^*}{h}\right) &= \frac{6}{5h}(S_{44}V_y+S_{45}V_x)-\frac{S_{44}}{10}(s_1+s_2)-\frac{S_{45}}{10}(t_1+t_2)\\
\frac{3}{4}\left(\bar{w}_{,x}-\hat{w}_{,x}+\frac{4u^*}{h}\right) &= \frac{6}{5h}(S_{45}V_y+S_{55}V_x)-\frac{S_{45}}{10}(s_1+s_2)-\frac{S_{55}}{10}(t_1+t_2).
\end{aligned}
\tag{17}
$$

Equilibrium equations

$$
\begin{aligned}
&N_{x,x}+N_{xy,y}+t_2-t_1=0\\
&N_{xy,x}+N_{y,y}+s_2-s_1=0\\
&V_{x,x}+V_{y,y}+\frac{20M_z}{h^2}+p_1-p_2-\frac{h}{6}(t_{1,x}+t_{2,x}+s_{1,y}+s_{2,y})=0\\
&M_{x,x}+M_{xy,y}-V_x+\frac{h}{2}(t_1+t_2)=0\\
&M_{xy,x}+M_{y,y}-V_y+\frac{h}{2}(s_1+s_2)=0\\
&N_z-\frac{(p_1+p_2)h}{2}+\frac{h^2}{12}(t_{1,x}-t_{2,x}+s_{1,y}-s_{2,y})=0\\
&V_{x,x}+V_{y,y}+\frac{60M_z}{h^2}+5(p_1-p_2)-\frac{h}{2}(t_{1,x}+t_{2,x}+s_{1,y}+s_{2,y})=0.
\end{aligned}
\tag{18}
$$

Interface relations depend upon the nature of the prescribed conditions on the interfacial planes, i.e. continuity or prescribed tractions and/or displacements may be specified. The latter conditions occur in the case of a cracked interfacial region.

Interface conditions

(a) Continuity ($k = 1, 2, \ldots N-1$)

$$
\begin{aligned}
&\overset{(k)}{t_2} = \overset{(k+1)}{t_1}\\
&\overset{(k)}{s_2} = \overset{(k+1)}{s_1}\\
&\overset{(k)}{p_2} = \overset{(k+1)}{p_1}\\
&\overset{(k)}{\beta_4} - \overset{(k)}{S_{44}}\overset{(k)}{T_4} - \overset{(k)}{S_{45}}\overset{(k)}{T_5} + \overset{(k+1)}{\alpha_4} - \overset{(k+1)}{S_{44}}\overset{(k+1)}{Q_4} - \overset{(k+1)}{S_{45}}\overset{(k+1)}{Q_5} = 0\\
&\overset{(k)}{\beta_5} - \overset{(k)}{S_{45}}\overset{(k)}{T_4} - \overset{(k)}{S_{55}}\overset{(k)}{T_5} + \overset{(k+1)}{\alpha_5} - \overset{(k+1)}{S_{45}}\overset{(k+1)}{Q_4} - \overset{(k+1)}{S_{55}}\overset{(k+1)}{Q_5} = 0\\
&\overset{(k)}{\gamma_2} - \overset{(k)}{S_{33}}\overset{(k)}{R_2} + \overset{(k+1)}{\gamma_1} - \overset{(k+1)}{S_{33}}\overset{(k+1)}{R_1} = 0.
\end{aligned}
\tag{19}
$$

(b) Prescribed Tractions and/or Displacements ($k = 1, 2 \ldots N-1$)

$$
\begin{aligned}
&\overset{(k)}{t_2} = \overset{(k)}{\tilde{t}_2} \text{ or } && \overset{(k)}{\beta_5} - \overset{(k)}{S_{45}}\overset{(k)}{T_4} - \overset{(k)}{S_{55}}\overset{(k)}{T_5} = -\overset{(k)}{\tilde{u}_2}\\
&\overset{(k)}{s_2} = \overset{(k)}{\tilde{s}_2} \text{ or } && \overset{(k)}{\beta_4} - \overset{(k)}{S_{44}}\overset{(k)}{T_4} - \overset{(k)}{S_{45}}\overset{(k)}{T_5} = -\overset{(k)}{\tilde{v}_2}\\
&\overset{(k)}{p_2} = \overset{(k)}{\tilde{p}_2} \text{ or } && \overset{(k)}{\gamma_2} - \overset{(k)}{S_{33}}\overset{(k)}{R_2} = -\overset{(k)}{\tilde{w}}\\
&\overset{(k+1)}{t_1} = \overset{(k+1)}{\tilde{t}} \text{ or } && \overset{(k+1)}{\alpha_5} - \overset{(k+1)}{S_{45}}\overset{(k+1)}{Q_4} - \overset{(k+1)}{S_{55}}\overset{(k+1)}{Q_5} = \overset{(k+1)}{\tilde{u}_1}\\
&\overset{(k+1)}{s_1} = \overset{(k+1)}{\tilde{s}_1} \text{ or } && \overset{(k+1)}{\alpha_4} - \overset{(k+1)}{S_{44}}\overset{(k+1)}{Q_4} - \overset{(k+1)}{S_{45}}\overset{(k+1)}{Q_5} = \overset{(k+1)}{\tilde{v}_1}\\
&\overset{(k+1)}{p_1} = \overset{(k+1)}{\tilde{p}_1} \text{ or } && \overset{(k+1)}{\gamma_1} - \overset{(k+1)}{S_{33}}\overset{(k+1)}{R_1} = \overset{(k+1)}{\tilde{w}_1}
\end{aligned}
\tag{20}
$$

where eqns (20) are to be understood in the sense that, at each interface, any combination which contains one equation from each line can be used to represent the interface boundary conditions in any region of the interface for which (19) are not prescribed.

Finally, the boundary conditions on the external surfaces of the body are given by:

Boundary conditions

(*a*) *Edge surface.* For the edge surface, one term from each of the following products must be prescribed *for each layer* (superscripts k are omitted)

$$
N_n\bar{u}_n,\ N_{ns}\bar{u}_s,\ M_n u_n^*,\ M_{ns} u_s^*,\ \left(\frac{3V_n}{h} - \frac{\tau_1 + \tau_2}{2}\right)\bar{w},
$$

$$
(\tau_2 - \tau_1)\, w^*,\ \left(\tau_1 + \tau_2 - \frac{2V_n}{h}\right)\hat{w}. \tag{21}
$$

(*b*) *Top surface.* The boundary conditions on the top surface are the same as the first three lines of (20) with $k = N$.

(*c*) *Bottom surface.* The boundary conditions on the bottom surface are the same as the last three lines of (20) with $k = 0$.

This completes the development of the present theory. We observe that the governing eqns (17)–(20), plus the boundary conditions on the top and bottom surfaces, continue a system of

$23N$ equations in terms of a like number of unknowns. The system can be reduced to $13N$ equations by solving (17) for the force and moment resultants in terms of the weighted displacement functions and interlaminar stress components and substituting into the remaining equations. Inspection of the governing equations reveals that interfacial displacements will only appear in the form of *prescribed* functions, hence they are not to be considered as dependent variables in the present theory. From (21), we see that $7N$ edge conditions are required in this theory. In the event that only edge tractions are prescribed in a given boundary value problem, these $7N$ edge functions may be taken to be the $3N$ force resultants, $2N$ moment resultants and $2N$ interlaminar shear stresses at the top and bottom of every layer.

Clearly, the requirements established in the introductory section are all satisfied by the present theory, in particular, the principle of "layer equilibrium". Furthermore, the generality of the interface conditions, (19) and (20), allow for the presence of interfacial cracks in the treatment of specific boundary value problems. Finally, the usual (physically meaningful) equations of equilibrium are represented by the first, second, fourth, and fifth of (18), along with a linear combination of the third and seventh of (18).

COMPARISON WITH FINITE ELEMENT RESULTS

In this section we shall relate the response predicted by the present theory to that given by numerical elasticity solutions for several problems of practical and theoretical interest. We shall treat the class of boundary value problems known as the free edge problem in which a laminate of finite width is subjected to a uniform axial strain $\epsilon_x = \epsilon$[4]. The origin of coordinates is located at the center of the laminate and each layer is reinforced by a system of parallel fibers oriented at an angle θ with the x-axis as shown in Fig. 1. The fibers in the various layers all lie in planes parallel to xy, and the laminate is symmetric, i.e. $\theta(z) = \theta(-z)$. In the analysis of the stress field, which is only a function of y and z, each layer is treated as a homogeneous, anisotropic body represented by its effective moduli and stresses will be denoted by functions of the form $f(y, z)$.

Comprehensive results based upon the finite element method have recently been presented by Wang and Crossman[17] for this class of boundary value problems in laminate elasticity. Hence, that work will be employed here to compare specific results given by the present theory. Two particular laminates; [0, 90], in which the values of θ in consecutive layers are 0°, 90°, 90°, 0°, and [±45], in which the orientations are 45°, −45°, −45°, 45°, will be examined in this study. The layers are of equal thickness h, the laminate width is $2b = 16h$, and the moduli in the planes

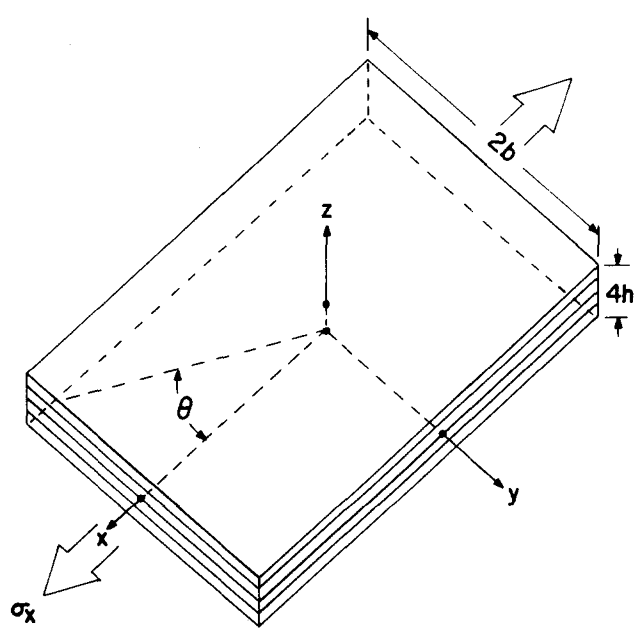

Fig. 1. Laminate geometry.

of elastic symmetry of each layer are given by

$$E_{11} = 20 \times 10^6 \text{ psi}, \quad E_{22} = E_{33} = 2.1 \times 10^6 \text{ psi}$$
$$G_{12} = G_{13} = G_{23} = 0.85 \times 10^6 \text{ psi}$$
$$\nu_{12} = \nu_{13} = \nu_{23} = 0.21$$

where 1, 2 and 3 refer to the fiber, transverse, and thickness directions, respectively, and ν_{12}, for example, is the Poisson ratio measuring strain in the transverse direction due to uniaxial tension in the fiber direction.

In Figs. 2–5, we compare various features of the response for the [±45] laminate as given by the present theory[35] and the finite element solution of[17]. The values of N in these figures correspond to the number of sub-layers used in the present theory to model one-half of the laminate. Thus, $N = 6$ implies that each physical layer of thickness h in the body has been modeled by three sub-layers, each of thickness $h/3$, while $N = 2$ indicates that each physical layer is treated as a unit.

In Figs. 2 and 3 are shown the distribution of σ_x and τ_{xy}, respectively, along the width of the laminate at the center of the top (physical) layer. The functions given by the present theory were computed via eqns (8). The results for $N = 6$ and the finite element solution are nearly coincident for all values of y, while the $N = 2$ results differ by only a few percent in the boundary layer region.

Even the $N = 2$ result agrees quite well with that of the finite element solution for the width distribution of τ_{xz} at the ±45° interface. However, a singularity is expected at this level at the free edge[4, 17]. The presence of a singularity introduces some ambiguity in the finite element

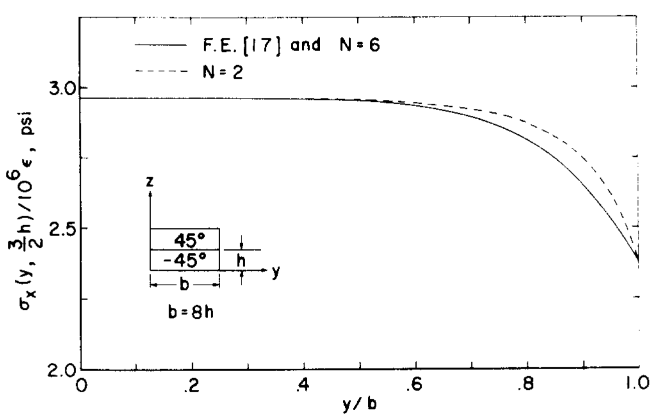

Fig. 2. Distribution of σ_x along center of top layer ($z = (3/2)h$).

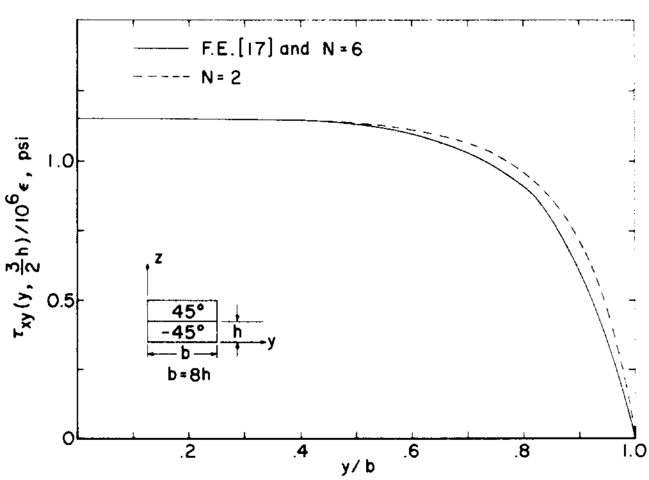

Fig. 3. Distribution of τ_{xy} along center of top layer ($z = (3/2)h$).

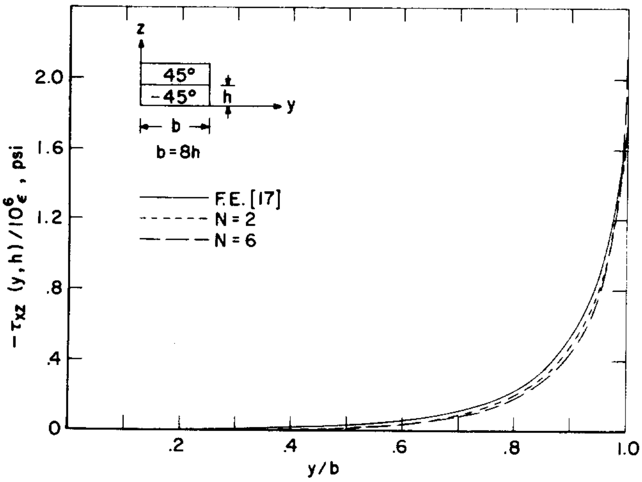

Fig. 4. Distribution of τ_{xz} along ±45° interface ($z = h$).

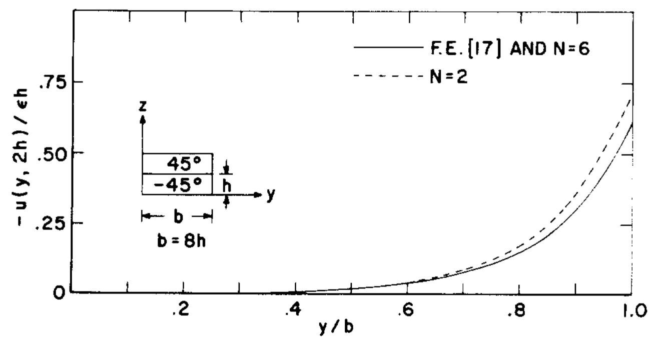

Fig. 5. Axial desplacement across top surface ($z = 2h$).

solution, so that the curve given here involves some judgement in the interpretation of the numerical results. On the other hand, the singularity is manifested in a different way in the present theory, i.e. the stress component τ_{xz} at the singular point grows with increasing values of N. Whether a finite limit is approached for large N has not been established. This situation is similar to the rise in average stress in the element adjacent to the singular point as the element size decreases in the finite element method. However, the present theory contains no singularity (see [35]), consequently the computed stress distribution is an *exact solution* in this formulation. The growth of stress component $\tau_{xz}(b, h)$ with N is shown in Table 1. The result for $N = 3$ represents the average given by the case in which the lower layer is represented by 2 sub-layers and the upper layer by one, and the opposite situation, although the two results are nearly identical. The same interpretation is invoked for $N = 5$. Unfortunately, because of the magnitudes of the numbers involved in the solution approach employed in [35], values of N larger than 6 could not be considered. Clearly, the manner in which singular behavior is portrayed in the present theory needs further study. In particular, the approach by which one correlates the analytical results with delamination failure tests needs consideration.

Table 1. Growth of maximum stress with N in [±45]

N	τ_{xz} (b,h) / 10^6 ε (psi)
2	1.664
3	1.798
4	2.017
5	2.102
6	2.213

Although displacement components are not dependent variables in the present theory, the weighted displacement functions can be used to approximate them with the aid of an assumed variation within each layer. For example, if we assume that axial displacement u is a linear function of z within each layer, by use of (14) we can show that

$$u_k = \frac{\bar{u}_k}{2} + \frac{3u_k^*}{h_k} z_k \quad (22)$$

where z_k is measured from a local coordinate system at the center of the kth layer. Agreement between this approach and the finite element result for axial displacement distribution across the width of the top surface is quite good as shown in Fig. 5.

In Fig. 6, the distribution of σ_z along the width direction on the central plane ($z = 0$) of the [0, 90] laminate is shown. Clearly, the present theory with $N = 6$ agrees wuite well with the finite element result, while the $N = 2$ result appears accurate except in a region very close to the free edge. Not shown on the figure is the result for $N = 4$, which has a very slight hump near the free edge and attains a maximum value close to the $N = 6$ result.

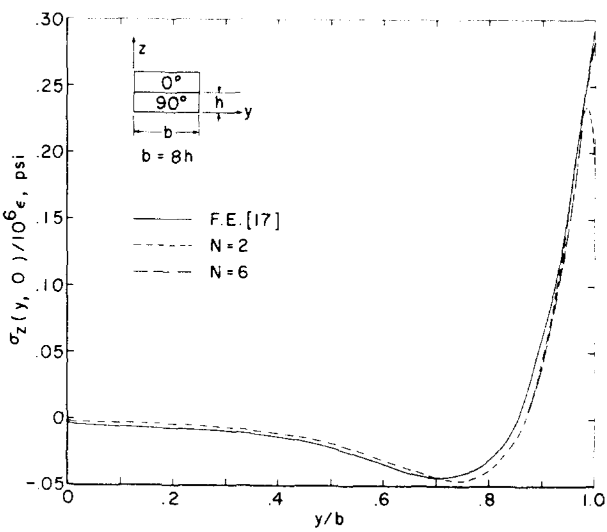

Fig. 6. Distribution of σ_z along central plane ($z = 0$).

Figure 7 illustrates the width-dependence of σ_z at the interface between the 0° and 90° layers, where in contrast to Fig. 6, a singularity is expected at the free edge owing to the discontinuity in elastic properties. The finite element solution gives strong evidence of the singularity since extreme variability occurs in the neighborhood of (b, h). Because of this, the finite element results are somewhat subjective in this region. Again, the $N = 6$ result is closer to the finite element curve than that of $N = 2$. As before (Fig. 4), the present theory yields finite maximum stresses which appear to grow monotonically with increasing N at the singular point.

Comparative results for the distribution of τ_{yz} at the 0°–90° interface are shown in Fig. 8. The present theory satisfies the traction-free boundary condition, however, whether the finite element solution, or indeed, an exact elasticity solution, satisfies this condition (see [36]) is not known. However, generally reasonable agreement can be observed. According to the new theory, it appears that the function is approaching a finite peak value, although we cannot be certain until the solution for larger values of N is determined.

Variation of transverse displacement v at the top surface is shown in Fig. 9. The values in the present theory were defined by approximating the layer displacement as a linear function of z, which leads to an equation of the same form as (22). Excellent agreement is seen to occur between the two solution techniques.

An extremely steep stress gradient at an (apparent) singularity in σ_z was reported by Rybicki and Pagano [37] for a free edge problem in which one layer was isotropic. Using the

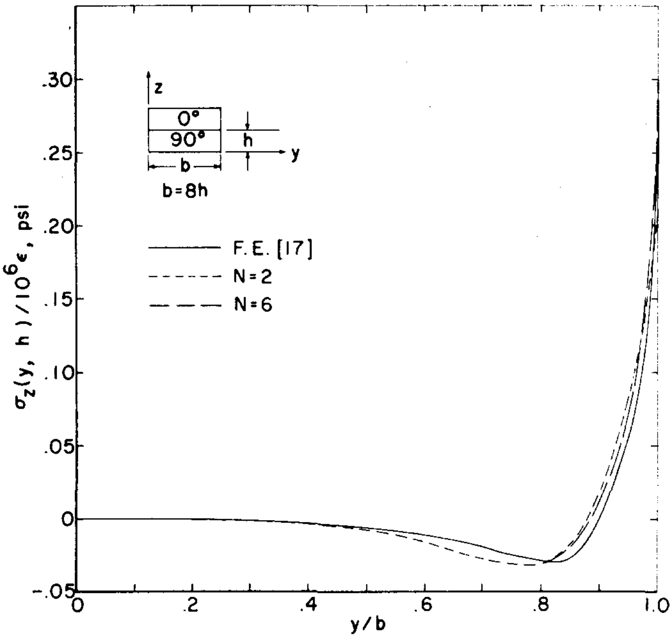

Fig. 7. Distribution of σ_z along 0/90 interface ($z = h$).

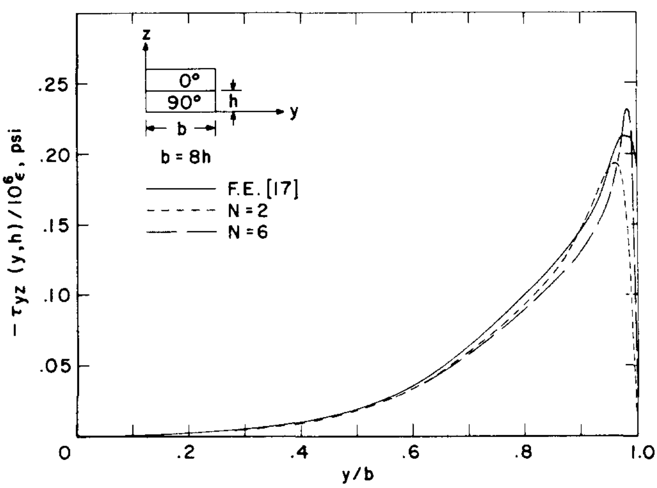

Fig. 8. Distribution of τ_{yz} along 0/90 interface ($z = h$).

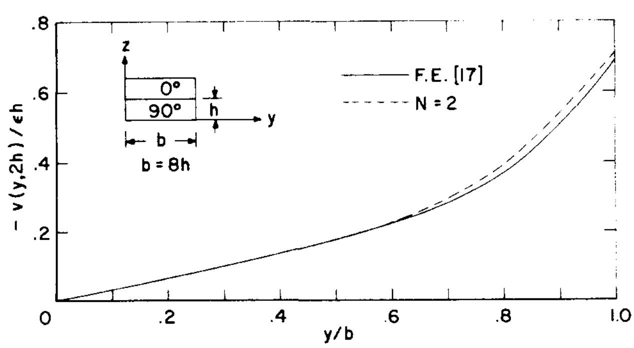

Fig. 9. Transverse displacement across top surface ($z = 2h$).

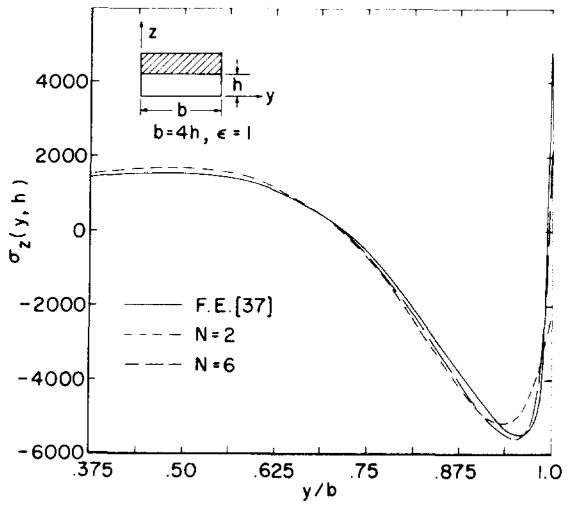

Fig. 10. Distribution of σ_z along interface ($z = h$).

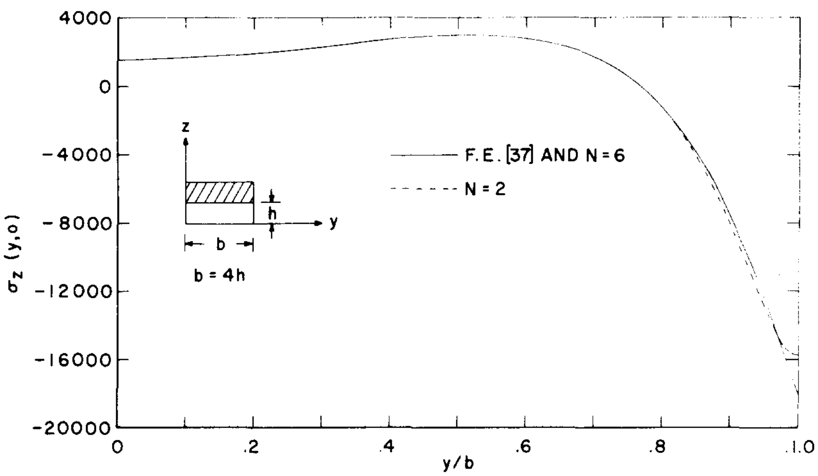

Fig. 11. Distribution of σ_z along central plane ($z = 0$).

moduli given in[37], comparative results are shown for the distributions of σ_z at the interface (Fig. 10) and central plane (Fig. 11), where comparable agreement with previous results can be observed.

CONCLUDING REMARKS

We have derived an approximate theory for the stress analysis of laminated bodies which resolves the difficulties involved in previous theories based upon assumed displacement fields. This theory is based upon Reissner's variational principle and assumed in-plane stresses that are linear functions of thickness coordinate z within each layer. While the appearance of $13N$ field equations and $7N$ edge conditions may seem to be overly cumbersome in actual problem solving, this level of detail is *required* to compute realistic global stress fields. The present theory guarantees satisfaction of "layer equilibrium" and allows the prescription of combinations of interfacial tractions and displacements which permit treatment of such conditions as interfacial continuity or cracks.

Comparison with existing solutions of the laminate free-edge class of boundary value problems, in which very steep stress gradients occur, has led to encouraging results. Although certain highly localized details of the stress field have been expunged when each layer was modeled as a single unit, this approach may be adequate for purposes of structural design. If this is not the case, based on the present study, the introduction of two or three sub-layers will produce dramatic improvements in accuracy. Alternatively, one may incorporate higher order terms in z into eqns (8) to develop more accurate theories satisfying the basic requirements set forth here. Such theories may eliminate the need for the use of sub-layers, but will obviously lead to greater complexity in the solutions of specific boundary value problems.

The situation regarding singularities remains somewhat nebulous since the precise nature of the singular stress field in the vicinity of an interface at an edge has not been established. We can state however, that the finite element solution can be severely hampered by the presence of elastic stress singularities, and stress field determination in their vicinity may become quite subjective. On the other hand, the present theory contains no edge singularities, an advantage from the problem solving viewpoint, however, examples have demonstrated a tendency for the computed maximum stress to grow with decreasing sub-layer thickness. Thus, problem solving has become simplified, but a method to *interpret* the stress predictions needs to be developed. We should notice, however, that the singularities given in effective modulus theories are mathematical artifacts in the treatment of fiber reinforced laminated bodies. This has been discussed in[37, 38], where support was given to the use of integrated stresses rather than point stresses in failure analysis. This point, together with the automatic satisfaction of layer equilibrium, as well as the capability for objective determination of laminate stress fields, favor the use of the present theory over approaches based upon numerical solutions of the elasticity equations. Unfortunately, this work, along with that of[35] and[17], demonstrate the extreme difficulties associated with attempts to realistically define the stress fields in laminates consisting of very many layers.

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